MATH 303 – Measure Theory
Homework 5

Instructions: Please upload a pdf of your solutions by 23:59 on Monday, ~~November 17~~ November 24. The assignment will be graded out of 10 points, taking into account both correctness and quality of presentation. More details on grading, as well as guidelines for mathematical writing, can be found on Moodle.

This problem addresses an application of the Riesz Representation Theorem. As with many applications of this theorem, what we really use is the induced topology (the *vague topology*) on the space of measures. It will be helpful to look at Section 5.5 from the lecture notes before attempting this problem (specifically, see the remark on the top of p. 72 for a description of the vague topology and the two results on p. 73).

A function $a : \mathbb{Z} \to \mathbb{C}$ is called *positive definite* if for every function $c : \mathbb{Z} \to \mathbb{C}$ with finite support, one has

$$\sum_{n,m\in\mathbb{Z}} a(n-m)c(m)\overline{c(n)} \geq 0.$$

(Equivalently, for every $N \in \mathbb{N}$, the matrix

$$\begin{pmatrix} a(0) & a(-1) & \dots & a(-N) \\ a(1) & a(0) & \dots & a(-N+1) \\ \vdots & \vdots & \ddots & \vdots \\ a(N) & a(N-1) & \dots & a(0) \end{pmatrix}$$

is positive semi-definite.)

Given a finite Borel measure μ on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$[1], we define the *Fourier transform* $\hat{\mu} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{\mu}(n) = \int_0^1 e^{-2\pi i n x}\ d\mu(x).$$

(a) Let μ be a finite Borel measure on $\mathbb{T}$. Show that $\hat{\mu}$ is positive definite.

(b) Let $a : \mathbb{Z} \to \mathbb{C}$ be positive definite. Show that there exists a finite Borel measure μ on $\mathbb{T}$ such that $a = \hat{\mu}$ as follows.

For $n \geq 0$, let $S_n : \mathbb{T} \to \mathbb{C}$ be the trigonometric polynomial $S_n(x) = \sum_{k=-n}^{n} a(k)e^{2\pi i k x}$. Then let $f_N = \frac{1}{N}\sum_{n=0}^{N-1} S_n$ for $N \in \mathbb{N}$.

(i) Show that

$$f_N(x) = \sum_{k=-N}^{N} a(k)e^{2\pi i k x}\left(1 - \frac{|k|}{N}\right) = \frac{1}{N}\sum_{n,m=0}^{N-1} a(n-m)e^{2\pi i (n-m)x}.$$

(ii) Use (i) to show that μ_N defined by $\mu_N(E) = \int_E f_N\ d\lambda$ for $E \in \mathrm{Borel}(\mathbb{T})$ is a measure with $\mu_N(\mathbb{T}) = a(0)$, and show that we can take μ as a vague limit point of $(\mu_N)_{N\in\mathbb{N}}$.

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[1] We express elements of $\mathbb{T}$ as numbers in $[0, 1)$, but $\mathbb{T}$ has the advantage of being a compact group.

Solution: **(a)** Let $c : \mathbb{Z} \to \mathbb{C}$ be an arbitrary function with finite support. Then using linearity of the integral, we have

$$\sum_{n,m\in\mathbb{Z}} \hat{\mu}(n-m)c(m)\overline{c(n)} = \int_0^1 \sum_{n,m\in\mathbb{Z}} e^{-2\pi i(n-m)x}c(m)\overline{c(n)}\ d\mu(x)$$

$$= \int_0^1 \left|\sum_{m\in\mathbb{Z}} e^{2\pi imx}c(m)\right|^2\ d\mu(x) \geq 0.$$

Thus, $\hat{\mu}$ is positive definite.

(b) **(i)** Unpacking the definition and swapping the order of summation, we have

$$f_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} S_n(x) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} a(k)e^{2\pi ikx} = \frac{1}{N}\sum_{k=-N}^{N}\underbrace{\sum_{n=0}^{N-1}\mathbb{1}\{|k|\leq n\}}_{N-|k|}a(k)e^{2\pi ikx}.$$

Distributing the $\frac{1}{N}$, we arrive at the first expression

$$f_N(x) = \sum_{k=-N}^{N} a(k)e^{2\pi ikx}\left(1-\frac{|k|}{N}\right).$$

For the second equality, we group terms by the value of $n-m$:

$$\frac{1}{N}\sum_{n,m=0}^{N-1} a(n-m)e^{2\pi i(n-m)x} = \frac{1}{N}\sum_{k=-N}^{N}\underbrace{\#\left\{(n,m)\in\{0,\ldots,N-1\}^2 : n-m=k\right\}}_{N-|k|}a(k)e^{2\pi ikx},$$

arriving once again at the same expression on the right-hand side.

(ii) The expression

$$\sum_{n,m=0}^{N-1} a(n-m)e^{2\pi i(n-m)x}$$

is of the form

$$\sum_{n,m\in\mathbb{Z}} a(n-m)c(m)\overline{c(n)}$$

for

$$c(n) = \begin{cases} e^{-2\pi inx}, & \text{if } 0\leq n\leq N-1\\ 0, & \text{otherwise.}\end{cases}$$

Since a is positive definite, we conclude that $f_N(x)\geq 0$ for every $x\in\mathbb{T}$. Therefore, μ_N takes nonnegative values.

Let us check that μ_N is a measure. First, $\mu_N(\emptyset) = \int_{\mathbb{T}} f \cdot \mathbb{1}_{\emptyset} \, d\lambda = \int_{\mathbb{T}} 0 \, d\lambda = 0$. Now suppose $(E_n)_{n\in\mathbb{N}}$ is a sequence of disjoint Borel subsets of $\mathbb{T}$. Then

$$\mu_N\left(\bigsqcup_{n\in\mathbb{N}} E_n\right) = \int_{\mathbb{T}} f_N \cdot \mathbb{1}_{\bigsqcup_{n\in\mathbb{N}} E_n} \, d\lambda = \int_{\mathbb{T}} f_N \cdot \sum_{n=1}^{\infty} \mathbb{1}_{E_n} \, d\lambda.$$

By linearity and the monotone convergence theorem, we can interchange the integral and sum (see Theorem 3.12 from the lecture notes), so

$$\mu_N\left(\bigsqcup_{n\in\mathbb{N}} E_n\right) = \sum_{n=1}^{\infty} \int_{\mathbb{T}} f_N \cdot \mathbb{1}_{E_n} \, d\lambda = \sum_{n=1}^{\infty} \mu_N(E_n).$$

As shown in the previous homework, the Lebesgue integral of the function $x \mapsto e^{2\pi inx}$ on $[0,1)$ agrees with the Riemann integral, so we can apply the fundamental theorem of calculus to compute

$$\mu_N(\mathbb{T}) = \int_0^1 f_N(x) \, dx = \sum_{k=-N}^{N} a(k) \left(1 - \frac{|k|}{N}\right) \underbrace{\int_0^1 e^{2\pi ikx} \, dx}_{\mathbb{1}\{k=0\}} = a(0).$$

Since $\mathbb{T}$ is a compact set, the space of measures with total mass $a(0)$ is vaguely compact. Let μ be a vague limit point of the sequence $(\mu_N)_{N\in\mathbb{N}}$, say $\mu = \lim_{j\to\infty} \mu_{N_j}$.[a] Given $n \in \mathbb{Z}$, the function $x \mapsto e^{-2\pi inx}$ is a continuous function on $\mathbb{T}$, so by the definition of the vague topology,

$$\begin{aligned}\hat{\mu}(n) = \int_0^1 e^{-2\pi inx} \, d\mu(x) &= \lim_{j\to\infty} \int_0^1 e^{-2\pi inx} \, d\mu_{N_j}(x) \overset{(*)}{=} \lim_{j\to\infty} \int_0^1 e^{-2\pi inx} f_{N_j}(x) \, dx \\ &= \lim_{j\to\infty} \sum_{k=-N_j}^{N_j} a(k) \left(1 - \frac{|k|}{N_j}\right) \underbrace{\int_0^1 e^{2\pi i(k-n)x} \, dx}_{\mathbb{1}\{k=n\}} = \lim_{j\to\infty} a(n) \left(1 - \frac{|n|}{N_j}\right) = a(n)\end{aligned}$$

as desired. The equality $(*)$ may be justified by approximating $e^{-2\pi inx}$ by simple functions and applying the dominated convergence theorem.

[a] One can actually show that $(\mu_N)_{N\in\mathbb{N}}$ is a convergent sequence using the Stone–Weierstrass theorem, but this is not needed to construct the measure μ.